Axiomatizing Modal Logic over Semilattices

Søren Brinck Knudstorp
Extract from MSc thesis, supervised by Johan van Benthem and Nick Bezhanishvili

September 11, 2023

Universiteit van Amsterdam

Outline of the talk

- Introduction and motivation
- Informal presentation of key ideas going into the axiomatization
- Conclusion

Defining Modal Logic over Semilattices

Definition (language and semantics)

The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle\sup\rangle\varphi\psi,$$

and the semantics of '$\langle\sup\rangle$' is:

$$w \Vdash \langle\sup\rangle\varphi\psi \quad \text{iff} \quad \exists u, v(u \Vdash \varphi;\ v \Vdash \psi; \\ w = \sup\{u, v\})$$

Example

Definition (frames and logic)

- A semilattice frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a join-semilattice on W (i.e., re., tr., anti-symm. and w. all binary joins).
- The modal logic over semilattices is denoted MIL_{Sem} and defined as

$$MIL_{Sem} := \{\varphi \in \mathcal{L}_M \mid \varphi \text{ is valid on all semilattice frames } (W, \leq)\}.$$

Main concern of this talk: Axiomatizing this logic.

Defining Modal Logic over Semilattices

Definition (language and semantics)

The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \mathrm{sup} \rangle \varphi\psi,$$

and the semantics of '$\langle \mathrm{sup} \rangle$' is:

$$w \Vdash \langle \mathrm{sup} \rangle \varphi\psi \quad \text{iff} \quad \exists u, v(u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example

Definition (frames and logic)

- A semilattice frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a join-semilattice on W (i.e., re., tr., anti-symm. and w. all binary joins).
- The modal logic over semilattices is denoted $\mathit{MIL}_{\mathit{Sem}}$ and defined as

$$\mathit{MIL}_{\mathit{Sem}} := \{\varphi \in \mathcal{L}_M \mid \varphi \textit{ is valid on all semilattice frames } (W, \leq)\}.$$

Main concern of this talk: Axiomatizing this logic.

Defining Modal Logic over Semilattices

Definition (language and semantics)

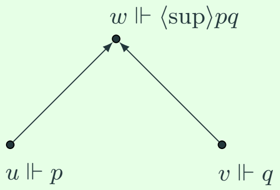
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example



Definition (frames and logic)

- A semilattice frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a join-semilattice on W (i.e., re., tr., anti-symm. and w. all binary joins).
- The modal logic over semilattices is denoted MIL_{Sem} and defined as

$$MIL_{Sem} := \{\varphi \in \mathcal{L}_M \mid \varphi \text{ is valid on all semilattice frames } (W, \leq)\}.$$

Main concern of this talk: Axiomatizing this logic.

Defining Modal Logic over Semilattices

Definition (language and semantics)

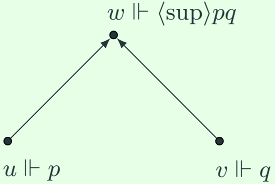
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example



Definition (frames and logic)

- A semilattice frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a join-semilattice on W (i.e., re., tr., anti-symm. and w. all binary joins).
- The modal logic over semilattices is denoted $\mathit{MIL}_{\mathit{Sem}}$ and defined as

$$\mathit{MIL}_{\mathit{Sem}} := \{\varphi \in \mathcal{L}_M \mid \varphi \text{ is valid on all semilattice frames } (W, \leq)\}.$$

Main concern of this talk: Axiomatizing this logic.

Defining Modal Logic over Semilattices

Definition (language and semantics)

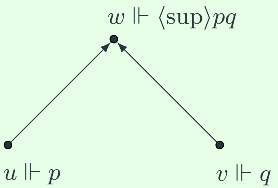
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example



Definition (frames and logic)

- A semilattice frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a join-semilattice on W (i.e., re., tr., anti-symm. and w. all binary joins).
- The modal logic over semilattices is denoted MIL_{Sem} and defined as

$$MIL_{Sem} := \{\varphi \in \mathcal{L}_M \mid \varphi \text{ is valid on all semilattice frames } (W, \leq)\}.$$

Main concern of this talk: Axiomatizing this logic.

Defining Modal Logic over Semilattices

Definition (language and semantics)

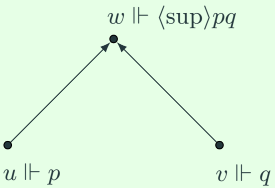
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v(u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example



Definition (frames and logic)

- A semilattice frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a join-semilattice on W (i.e., re., tr., anti-symm. and w. all binary joins).
- The modal logic over semilattices is denoted $\mathit{MIL}_{\mathit{Sem}}$ and defined as

$$\mathit{MIL}_{\mathit{Sem}} := \{\varphi \in \mathcal{L}_M \mid \varphi \textit{ is valid on all semilattice frames } (W, \leq)\}.$$

Main concern of this talk: Axiomatizing this logic.

Why semilattices?

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
    - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
    - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Motivation and objective

Why semilattices?

1. Lattices and groups were already taken [by Wang and Wang (2022), van Benthem and Bezhanishvili (2022), respectively]
2. Informational interpretation (hence the 'I' in 'MIL_{Sem}'):
   - Preorder and poset versions, MIL_{Pre} and MIL_{Pos}, introduced by van Benthem (1996)
   - Knudstorp (Forthcoming) proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable
3. Modal-logic analogue of truthmaker semantics

Why axiomatization?

Proof is tricky, but offers insights and additions to toolbox of techniques for (modal) completeness proofs; I hope to effectively communicate these ideas. Instead of presenting the completed proof as is, I'll go through the process of coming up with the proof/axiomatization.

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \text{sup} \rangle pq$

(4) $PPp \rightarrow Pp$

(Co.) $\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.) } \langle \text{sup} \rangle (\langle \text{sup} \rangle pq) r \leftrightarrow \langle \text{sup} \rangle p (\langle \text{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- **How to construct the satisfying model?**
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.)}\ \langle \text{sup} \rangle (\langle \text{sup} \rangle pq) r \leftrightarrow \langle \text{sup} \rangle p (\langle \text{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- **How to construct the satisfying model?**
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.)} \ \langle \sup \rangle (\langle \sup \rangle pq) r \leftrightarrow \langle \sup \rangle p (\langle \sup \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- **How to construct the satisfying model?**
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \mathrm{sup} \rangle pq$

(4) $PPp \rightarrow Pp$

(Co.) $\langle \mathrm{sup} \rangle pq \rightarrow \langle \mathrm{sup} \rangle qp$

(Dk.) $(p \wedge \langle \mathrm{sup} \rangle qr) \rightarrow \langle \mathrm{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.)} \ \langle \mathrm{sup} \rangle (\langle \mathrm{sup} \rangle pq) r \leftrightarrow \langle \mathrm{sup} \rangle p (\langle \mathrm{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- How to construct the satisfying model?
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \text{sup} \rangle pq$

(4) $PPp \rightarrow Pp$

(Co.) $\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.)}\ \langle \text{sup} \rangle (\langle \text{sup} \rangle pq) r \leftrightarrow \langle \text{sup} \rangle p (\langle \text{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- How to construct the satisfying model?
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.)} \ \langle \text{sup} \rangle (\langle \text{sup} \rangle pq) r \leftrightarrow \langle \text{sup} \rangle p (\langle \text{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- **How to construct the satisfying model?**
    - *Will the canonical model do?* No, not even close
    - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.) } \langle \text{sup} \rangle (\langle \text{sup} \rangle pq) r \leftrightarrow \langle \text{sup} \rangle p (\langle \text{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- **How to construct the satisfying model?**
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Starting point for axiomatizing MIL_{Sem}

Axiomatization of MIL_{Pre} [Knudstorp (Forthcoming)]

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \mathsf{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \mathsf{sup} \rangle pq \to \langle \mathsf{sup} \rangle qp$

(Dk.) $(p \wedge \langle \mathsf{sup} \rangle qr) \to \langle \mathsf{sup} \rangle pq$

- **Initial thought:** Since $MIL_{Pre} = MIL_{Pos}$, could it be that even $MIL_{Pre} = MIL_{Pos} = MIL_{Sem}$? No, as witnessed by

$$\text{(As.)} \ \langle \text{sup} \rangle (\langle \text{sup} \rangle pq) r \leftrightarrow \langle \text{sup} \rangle p (\langle \text{sup} \rangle qr).$$

- **First conclusion:** We must supplement with additional axioms.
- **Method for finding axioms:** We assume we have some MCS Γ_0 and work out what axioms are needed to construct a satisfying semilattice model.
- **How to construct the satisfying model?**
  - *Will the canonical model do?* No, not even close
  - *How about step-by-step?* Perhaps, let's try!

Step-by-step: idea

Step-by-step procedure:

(Base) Define singleton semilattice frame $\mathbb{F}_0 := (\{\{*\}\}, \{(\{*\}, \{*\})\})$ and 'label' it with our MCS: $l_0(\{*\}) = \Gamma_0$.

(Ind.) Step-wise construct $(\mathbb{F}_{n+1}, l_{n+1})$ from $(\mathbb{F}_n, l_n)$.
The goal being to prove a 'truth lemma' s.t. (after all finite steps)

$$(\mathbb{F}_\omega, V), \{*\} \Vdash \varphi \quad \Leftrightarrow \quad \varphi \in l_\omega(\{*\}) = \Gamma_0.$$

This achieves the reduction:

Axiomatizing MIL$_{Sem}$ ⇝ Finding (sound) axioms enabling this construction.

Step-by-step procedure:

(Base) Define singleton semilattice frame $\mathbb{F}_0 := (\{\{*\}\}, \{(\{*\}, \{*\})\})$ and 'label' it with our MCS: $l_0(\{*\}) = \Gamma_0$.

(Ind.) Step-wise construct $(\mathbb{F}_{n+1}, l_{n+1})$ from $(\mathbb{F}_n, l_n)$.
The goal being to prove a 'truth lemma' s.t. (after all finite steps)

$$(\mathbb{F}_\omega, V), \{*\} \Vdash \varphi \quad \Leftrightarrow \quad \varphi \in l_\omega(\{*\}) = \Gamma_0.$$

This achieves the reduction:

Axiomatizing MIL_{Sem} $\rightsquigarrow$ Finding (sound) axioms enabling this construction.

Step-by-step: idea

Step-by-step procedure:

(Base) Define singleton semilattice frame $\mathbb{F}_0 := (\{\{*\}\}, \{(\{*\}, \{*\})\})$ and 'label' it with our MCS: $l_0(\{*\}) = \Gamma_0$.

(Ind.) Step-wise construct $(\mathbb{F}_{n+1}, l_{n+1})$ from $(\mathbb{F}_n, l_n)$.
The goal being to prove a 'truth lemma' s.t. (after all finite steps)

$$(\mathbb{F}_\omega, V), \{*\} \Vdash \varphi \quad \Leftrightarrow \quad \varphi \in l_\omega(\{*\}) = \Gamma_0.$$

This achieves the reduction:

Axiomatizing MIL_{Sem} $\rightsquigarrow$ Finding (sound) axioms enabling this construction.

Step-by-step procedure:

(Base) Define singleton semilattice frame $\mathbb{F}_0 := (\{\{*\}\}, \{(\{*\},\{*\})\})$ and 'label' it with our MCS: $l_0(\{*\}) = \Gamma_0$.

(Ind.) Step-wise construct $(\mathbb{F}_{n+1}, l_{n+1})$ from $(\mathbb{F}_n, l_n)$. The goal being to prove a 'truth lemma' s.t. (after all finite steps)

$$(\mathbb{F}_\omega, V), \{*\} \Vdash \varphi \quad \Leftrightarrow \quad \varphi \in l_\omega(\{*\}) = \Gamma_0.$$

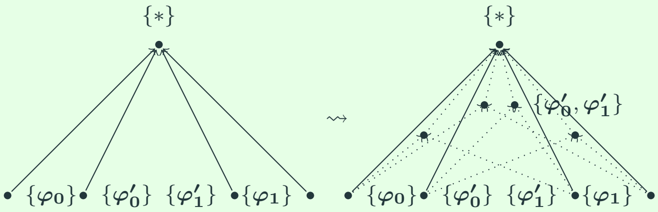
This achieves the reduction:

Axiomatizing MIL_{Sem} $\rightsquigarrow$ *Finding (sound) axioms enabling this construction.*

Step-by-step: obstacle 1

- Suppose $\{\langle\sup\rangle\varphi_0\varphi_0', \langle\sup\rangle\varphi_1\varphi_1'\} \subseteq l(\{*\}) = \Gamma_0$. Then add points $\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi_1'}\}$, and label them using the existence lemma (EL) s.t. $\varphi_0 \in l(\{\boldsymbol{\varphi_0}\})$, etc.
- *Problem:* Now $\{*\} = \sup\{\{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1'}\}\}$, but we need not have $l(\{*\})R_{\mathbf{Sem}}l(\{\boldsymbol{\varphi_0'}\})l(\{\boldsymbol{\varphi_1'}\})$, where '$R_{\mathbf{Sem}}$' is ternary relation of can. model.
- *Solution:* Add axiom $\pi_1 \in MIL_{Sem}$ enabling us to add a point, $\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}$, and label it s.t.

$$l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})R_{\mathbf{Sem}}l(\{\boldsymbol{\varphi_0'}\})l(\{\boldsymbol{\varphi_1'}\}) \quad \textit{and} \quad l(\{*\})R_{\mathbf{Sem}}l(\{\boldsymbol{\varphi_0}\})l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}), \text{ etc.}$$

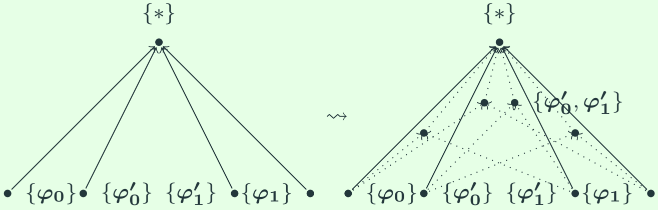


Obs: $\mathbb{M}, w \Vdash \langle\sup\rangle\varphi_0\varphi_0' \wedge \langle\sup\rangle\varphi_1\varphi_1' \quad \not\Rightarrow$ sub-semilattice is *isomorphic* to RHS, but $\mathbb{M}, w \Vdash \langle\sup\rangle\varphi_0\varphi_0' \wedge \langle\sup\rangle\varphi_1\varphi_1' \quad \Rightarrow$ sub-semilattice is *hom. im.* of RHS.

Step-by-step: obstacle 1

- Suppose $\{\langle\mathrm{sup}\rangle\varphi_0\varphi_0', \langle\mathrm{sup}\rangle\varphi_1\varphi_1'\} \subseteq l(\{*\}) = \Gamma_0$. Then add points $\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi_1'}\}$, and label them using the existence lemma (EL) s.t. $\varphi_0 \in l(\{\boldsymbol{\varphi_0}\})$, etc.
- *Problem:* Now $\{*\} = \mathrm{sup}\{\{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1'}\}\}$, but we need not have $l(\{*\})R_{\mathbf{Sem}}l(\{\boldsymbol{\varphi_0'}\})l(\{\boldsymbol{\varphi_1'}\})$, where '$R_{\mathbf{Sem}}$' is ternary relation of can. model.
- *Solution:* Add axiom $\pi_1 \in \mathit{MIL}_{\mathit{Sem}}$ enabling us to add a point, $\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}$, and label it s.t.

  $l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})R_{\mathbf{Sem}}l(\{\boldsymbol{\varphi_0'}\})l(\{\boldsymbol{\varphi_1'}\})$ *and* $l(\{*\})R_{\mathbf{Sem}}l(\{\boldsymbol{\varphi_0}\})l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})$, etc.

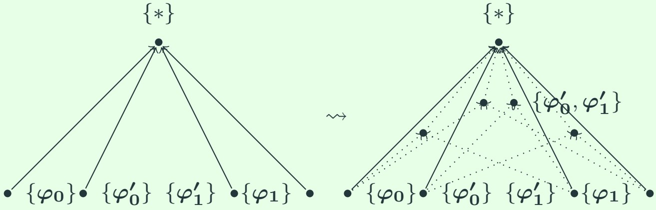


Obs: $\mathbb{M}, w \Vdash \langle\mathrm{sup}\rangle\varphi_0\varphi_0' \wedge \langle\mathrm{sup}\rangle\varphi_1\varphi_1' \quad \not\Rightarrow$ sub-semilattice is *isomorphic* to RHS, but $\mathbb{M}, w \Vdash \langle\mathrm{sup}\rangle\varphi_0\varphi_0' \wedge \langle\mathrm{sup}\rangle\varphi_1\varphi_1' \quad \Rightarrow$ sub-semilattice is *hom. im.* of RHS.

Step-by-step: obstacle 1

- Suppose $\{\langle \sup \rangle \varphi_0 \varphi'_0, \langle \sup \rangle \varphi_1 \varphi'_1\} \subseteq l(\{*\}) = \Gamma_0$. Then add points $\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi'_0}\}, \{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi'_1}\}$, and label them using the existence lemma (EL) s.t. $\varphi_0 \in l(\{\boldsymbol{\varphi_0}\})$, etc.
- *Problem:* Now $\{*\} = \sup\{\{\boldsymbol{\varphi'_0}\}, \{\boldsymbol{\varphi'_1}\}\}$, but we need not have $l(\{*\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\varphi'_0}\}) l(\{\boldsymbol{\varphi'_1}\})$, where '$R_{\mathbf{Sem}}$' is ternary relation of can. model.
- *Solution:* Add axiom $\pi_1 \in MIL_{Sem}$ enabling us to add a point, $\{\boldsymbol{\varphi'_0}, \boldsymbol{\varphi'_1}\}$, and label it s.t.

$$l(\{\boldsymbol{\varphi'_0}, \boldsymbol{\varphi'_1}\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\varphi'_0}\}) l(\{\boldsymbol{\varphi'_1}\}) \quad \textit{and} \quad l(\{*\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\varphi_0}\}) l(\{\boldsymbol{\varphi'_0}, \boldsymbol{\varphi'_1}\}), \text{ etc.}$$

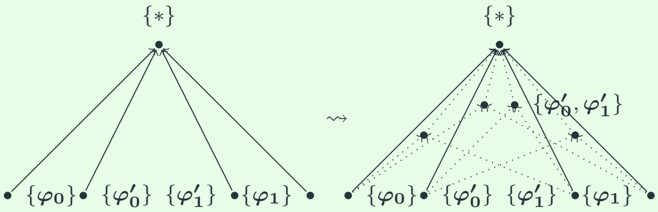


Obs: $\mathbb{M}, w \Vdash \langle \sup \rangle \varphi_0 \varphi'_0 \wedge \langle \sup \rangle \varphi_1 \varphi'_1 \quad \not\Rightarrow \quad$ sub-semilattice is *isomorphic* to RHS, but $\mathbb{M}, w \Vdash \langle \sup \rangle \varphi_0 \varphi'_0 \wedge \langle \sup \rangle \varphi_1 \varphi'_1 \quad \Rightarrow \quad$ sub-semilattice is *hom. im.* of RHS.

Step-by-step: obstacle 1

- Suppose $\{\langle \sup \rangle \varphi_0 \varphi_0', \langle \sup \rangle \varphi_1 \varphi_1'\} \subseteq l(\{*\}) = \Gamma_0$. Then add points $\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi_1'}\}$, and label them using the existence lemma (EL) s.t. $\varphi_0 \in l(\{\boldsymbol{\varphi_0}\})$, etc.
- *Problem:* Now $\{*\} = \sup\{\{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1'}\}\}$, but we need not have $l(\{*\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\varphi_0'}\}) l(\{\boldsymbol{\varphi_1'}\})$, where '$R_{\mathbf{Sem}}$' is ternary relation of can. model.
- *Solution:* Add axiom $\pi_1 \in$ *MIL*$_{Sem}$ enabling us to add a point, $\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}$, and label it s.t.

$$l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\varphi_0'}\}) l(\{\boldsymbol{\varphi_1'}\}) \quad \textit{and} \quad l(\{*\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\varphi_0}\}) l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}), \text{ etc.}$$



Obs: $\mathbb{M}, w \Vdash \langle \sup \rangle \varphi_0 \varphi_0' \wedge \langle \sup \rangle \varphi_1 \varphi_1' \quad \not\Rightarrow \quad$ sub-semilattice is *isomorphic* to RHS, but $\mathbb{M}, w \Vdash \langle \sup \rangle \varphi_0 \varphi_0' \wedge \langle \sup \rangle \varphi_1 \varphi_1' \quad \Rightarrow \quad$ sub-semilattice is *hom. im.* of RHS.

Step-by-step: obstacle 2

Takeaway:

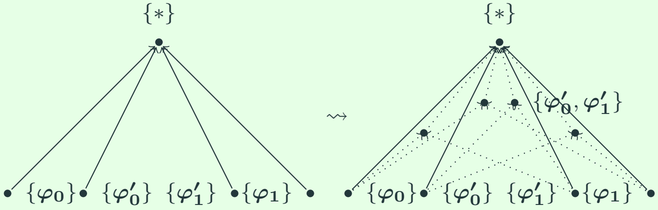
- Axioms, like π_1, are implications $\beta \to \alpha$ encoding:
  $\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements.

- Now suppose that $\langle \sup \rangle \psi \psi' \in l(\{\varphi_0\})$.
- *Problem*: adding $\{\psi\}, \{\psi'\}$ and labeling using EL for $l(\{\varphi_0\})$ does not work: then $\{*\} = \sup\{\{\psi'\}, \{\varphi'_0\}\}$ but maybe not $l(\{*\}) R_{\mathbf{Sem}} l(\{\psi'\}) l(\{\varphi'_0\})$.
- *Solution*: Adding axiom $\pi_2 \in MIL_{Sem}$ of greater 'depth', and use EL for $l(\{*\})$.

Step-by-step: obstacle 2

Takeaway:

- Axioms, like π_1, are implications $\beta \to \alpha$ encoding:
  $\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements.

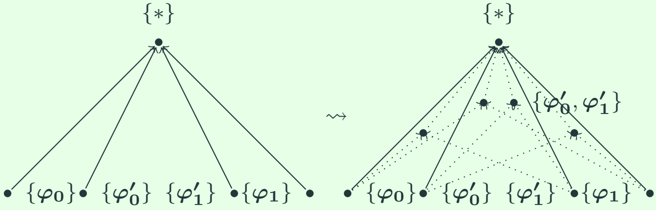


- Now suppose that $\langle \sup \rangle \psi \psi' \in l(\{\varphi_0\})$.
- *Problem:* adding $\{\psi\}, \{\psi'\}$ and labeling using EL for $l(\{\varphi_0\})$ does not work: then $\{*\} = \sup\{\{\psi'\}, \{\varphi_0'\}\}$ but maybe not $l(\{*\}) R_{\mathbf{Sem}} l(\{\psi'\}) l(\{\varphi_0'\})$.
- *Solution:* Adding axiom $\pi_2 \in MIL_{Sem}$ of greater 'depth', and use EL for $l(\{*\})$.

Step-by-step: obstacle 2

Takeaway:

- Axioms, like π_1, are implications $\beta \to \alpha$ encoding:
  $\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements.

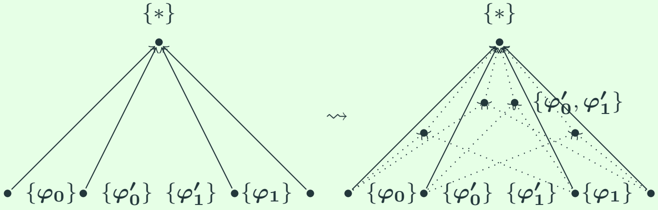


- Now suppose that $\langle \sup \rangle \psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$.
- *Problem:* adding $\{\psi\}, \{\psi'\}$ and labeling using EL for $l(\{\boldsymbol{\varphi_0}\})$ does not work: then $\{*\} = \sup\{\{\psi'\}, \{\boldsymbol{\varphi'_0}\}\}$ but maybe not $l(\{*\}) R_{\mathbf{Sem}} l(\{\psi'\}) l(\{\boldsymbol{\varphi'_0}\})$.
- *Solution:* Adding axiom $\pi_2 \in MIL_{Sem}$ of greater 'depth', and use EL for $l(\{*\})$.

Step-by-step: obstacle 2

Takeaway:

- Axioms, like π_1, are implications $\beta \to \alpha$ encoding:
  $\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements.

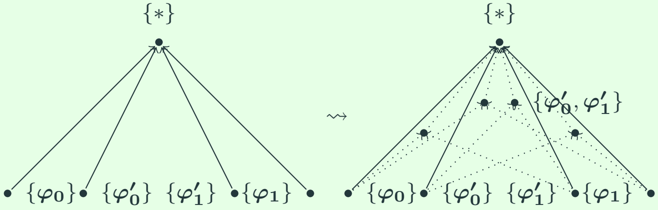


- Now suppose that $\langle \sup \rangle \psi \psi' \in l(\{\varphi_0\})$.
- *Problem:* adding $\{\psi\}, \{\psi'\}$ and labeling using EL for $l(\{\varphi_0\})$ does not work: then $\{*\} = \sup\{\{\psi'\}, \{\varphi_0'\}\}$ but maybe not $l(\{*\}) R_{\mathbf{Sem}} l(\{\psi'\}) l(\{\varphi_0'\})$.
- *Solution:* Adding axiom $\pi_2 \in MIL_{Sem}$ of greater 'depth', and use EL for $l(\{*\})$.

Step-by-step: obstacle 2

Takeaway:

- Axioms, like π_1, are implications $\beta \to \alpha$ encoding:
  $\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements.



- Now suppose that $\langle \sup \rangle \psi \psi' \in l(\{\boldsymbol{\varphi_0}\})$.
- *Problem:* adding $\{\boldsymbol{\psi}\}, \{\boldsymbol{\psi'}\}$ and labeling using EL for $l(\{\boldsymbol{\varphi_0}\})$ does not work: then $\{*\} = \sup\{\{\boldsymbol{\psi'}\}, \{\boldsymbol{\varphi_0'}\}\}$ but maybe not $l(\{*\}) R_{\mathbf{Sem}} l(\{\boldsymbol{\psi'}\}) l(\{\boldsymbol{\varphi_0'}\})$.
- *Solution:* Adding axiom $\pi_2 \in$ *MIL*$_{Sem}$ of greater 'depth', and use EL for $l(\{*\})$.

Step-by-step: obstacle 3

Takeaways:

- To achieve the truth lemma, we need formulas $\pi_1, \pi_2, \ldots$ of incr. depth;
- and $\pi_1, \pi_2, \ldots$ must be constructed so that they can be evaluated at the same MCS $l(\{*\}) = \Gamma_0$.

Problem: Having labeled, e.g., $\{\boldsymbol{\varphi_0}\}$ via evaluating the formula π_1 at Γ_0, we then relabel $\{\boldsymbol{\varphi_0}\}$ via evaluating π_2 at Γ_0. How do we ascertain that $l_2(\{\boldsymbol{\varphi_0}\}) = l_1(\{\boldsymbol{\varphi_0}\})$?

Observation: While an MCS Θ is equivalently defined as an infinite conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a finite conjunction $\widehat{\Theta_F}$.

Solution:

- Aim for weak completeness instead: Extend consistent formula φ to the least subformula-closed set $\Phi \ni \varphi$.
- Instead of labeling with MCSs Θ, we label with $(\Theta \cap \Phi)$. This labeling can be coded into the formulas $\pi_1, \pi_2, \ldots$ to ensure $l_n(x) = l_{n+1}(x)$.

Step-by-step: obstacle 3

Takeaways:

- To achieve the truth lemma, we need formulas $\pi_1, \pi_2, \ldots$ of incr. depth;
- and $\pi_1, \pi_2, \ldots$ must be constructed so that they can be evaluated at the same MCS $l(\{*\}) = \Gamma_0$.

Problem: Having labeled, e.g., $\{\boldsymbol{\varphi_0}\}$ via evaluating the formula π_1 at Γ_0, we then relabel $\{\boldsymbol{\varphi_0}\}$ via evaluating π_2 at Γ_0. How do we ascertain that $l_2(\{\boldsymbol{\varphi_0}\}) = l_1(\{\boldsymbol{\varphi_0}\})$?

Observation: While an MCS Θ is equivalently defined as an infinite conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a finite conjunction $\widehat{\Theta_F}$.

Solution:

- Aim for weak completeness instead: Extend consistent formula φ to the least subformula-closed set $\Phi \ni \varphi$.
- Instead of labeling with MCSs Θ, we label with $(\Theta \cap \Phi)$. This labeling can be coded into the formulas $\pi_1, \pi_2, \ldots$ to ensure $l_n(x) = l_{n+1}(x)$.

Step-by-step: obstacle 3

Takeaways:

- To achieve the truth lemma, we need formulas $\pi_1, \pi_2, \ldots$ of incr. depth;
- and $\pi_1, \pi_2, \ldots$ must be constructed so that they can be evaluated at the same MCS $l(\{*\}) = \Gamma_0$.

Problem: Having labeled, e.g., $\{\boldsymbol{\varphi_0}\}$ via evaluating the formula π_1 at Γ_0, we then relabel $\{\boldsymbol{\varphi_0}\}$ via evaluating π_2 at Γ_0. How do we ascertain that $l_2(\{\boldsymbol{\varphi_0}\}) = l_1(\{\boldsymbol{\varphi_0}\})$?

Observation: While an MCS Θ is equivalently defined as an infinite conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a finite conjunction $\widehat{\Theta_F}$.

Solution:

- Aim for weak completeness instead: Extend consistent formula φ to the least subformula-closed set $\Phi \ni \varphi$.
- Instead of labeling with MCSs Θ, we label with $(\Theta \cap \Phi)$. This labeling can be coded into the formulas $\pi_1, \pi_2, \ldots$ to ensure $l_n(x) = l_{n+1}(x)$.

Step-by-step: obstacle 3

Takeaways:

- To achieve the truth lemma, we need formulas $\pi_1, \pi_2, \ldots$ of incr. depth;
- and $\pi_1, \pi_2, \ldots$ must be constructed so that they can be evaluated at the same MCS $l(\{*\}) = \Gamma_0$.

Problem: Having labeled, e.g., $\{\varphi_{\mathbf{0}}\}$ via evaluating the formula π_1 at Γ_0, we then relabel $\{\varphi_{\mathbf{0}}\}$ via evaluating π_2 at Γ_0. How do we ascertain that $l_2(\{\varphi_{\mathbf{0}}\}) = l_1(\{\varphi_{\mathbf{0}}\})$?

Observation: While an MCS Θ is equivalently defined as an infinite conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a finite conjunction $\widehat{\Theta_F}$.

Solution:

- Aim for weak completeness instead: Extend consistent formula φ to the least subformula-closed set $\Phi \ni \varphi$.
- Instead of labeling with MCSs Θ, we label with $(\Theta \cap \Phi)$. This labeling can be coded into the formulas $\pi_1, \pi_2, \ldots$ to ensure $l_n(x) = l_{n+1}(x)$.

Step-by-step: obstacle 3

Takeaways:

- To achieve the truth lemma, we need formulas $\pi_1, \pi_2, \ldots$ of incr. depth;
- and $\pi_1, \pi_2, \ldots$ must be constructed so that they can be evaluated at the same MCS $l(\{*\}) = \Gamma_0$.

Problem: Having labeled, e.g., $\{\boldsymbol{\varphi_0}\}$ via evaluating the formula π_1 at Γ_0, we then relabel $\{\boldsymbol{\varphi_0}\}$ via evaluating π_2 at Γ_0. How do we ascertain that $l_2(\{\boldsymbol{\varphi_0}\}) = l_1(\{\boldsymbol{\varphi_0}\})$?

Observation: While an MCS Θ is equivalently defined as an infinite conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a finite conjunction $\widehat{\Theta_F}$.

Solution:

- Aim for weak completeness instead: Extend consistent formula φ to the least subformula-closed set $\Phi \ni \varphi$.
- Instead of labeling with MCSs Θ, we label with $(\Theta \cap \Phi)$. This labeling can be coded into the formulas $\pi_1, \pi_2, \ldots$ to ensure $l_n(x) = l_{n+1}(x)$.

Step-by-step: obstacle 4

Problem: How can π_i determine what Φ-formulas the witnessing worlds are to satisfy *and* yet be sound: that, say, some $w \Vdash \langle \mathrm{sup} \rangle \varphi_0 \varphi_0'$ does not determine what Φ-formulas the φ_0- and φ_0'-world satisfy.

Recall: "Axioms, like π_1, are implications $\beta \to \alpha$ encoding:

$\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements."

Solution: "Axioms, like π_1, are implications ~~$\beta \to \alpha$~~ [$\beta \to \bigvee \alpha$] encoding:

$\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of ~~a certain other~~ [one of the] semilattice[s] freely generated modulo some requirements [depending on the given disjunct, and where '$\bigvee$' quantifies over all possible 'Φ-names']"

Step-by-step: obstacle 4

Problem: How can π_i determine what Φ-formulas the witnessing worlds are to satisfy *and* yet be sound: that, say, some $w \Vdash \langle \mathrm{sup} \rangle \varphi_0 \varphi_0'$ does not determine what Φ-formulas the φ_0- and φ_0'-world satisfy.

Recall: "Axioms, like π_1, are implications $\beta \to \alpha$ encoding:

$\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements."

Solution: "Axioms, like π_1, are implications ~~$\beta \to \alpha$~~ $[\beta \to \bigvee \alpha]$ encoding:

$\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of ~~a certain other~~ [one of the] semilattice[s] freely generated modulo some requirements [depending on the given disjunct, and where '$\bigvee$' quantifies over all possible 'Φ-names']"

Step-by-step: obstacle 4

Problem: How can π_i determine what Φ-formulas the witnessing worlds are to satisfy *and* yet be sound: that, say, some $w \Vdash \langle \mathrm{sup} \rangle \varphi_0 \varphi_0'$ does not determine what Φ-formulas the φ_0- and φ_0'-world satisfy.

Recall: "Axioms, like π_1, are implications $\beta \to \alpha$ encoding:

$\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of a certain other semilattice freely generated modulo some requirements."

Solution: "Axioms, like π_1, are implications ~~$\beta \to \alpha$~~ $[\beta \to \bigvee \alpha]$ encoding:

$\mathbb{M}, w \Vdash \beta \quad \Rightarrow \quad$ 'witnessing sub-semilattice' is *hom. im.* of ~~a certain other~~ [one of the] semilattice[s] freely generated modulo some requirements [depending on the given disjunct, and where '$\bigvee$' quantifies over all possible 'Φ-names']"

Failure of (deterministic) step-by-step

Final problem: If the consequents of the formulas $\pi_1, \pi_2, \ldots$ consist of disjunctions defining *distinct* semilattices, which disjunct shall we choose when stepwise extending our semilattice as to satisfy the truth lemma?

Back to 'how to construct the satisfying semilattice model?'

- *Will the canonical model do?* ✗
- *Will 'deterministic' step-by-step do?* ✗
- *How about 'indeterministic' step-by-step?*

Failure of (deterministic) step-by-step

Final problem: If the consequents of the formulas $\pi_1, \pi_2, \ldots$ consist of disjunctions defining *distinct* semilattices, which disjunct shall we choose when stepwise extending our semilattice as to satisfy the truth lemma?

Back to 'how to construct the satisfying semilattice model?'

- *Will the canonical model do?* ✗
- *Will 'deterministic' step-by-step do?* ✗
- *How about 'indeterministic' step-by-step?*

Failure of (deterministic) step-by-step

Final problem: If the consequents of the formulas $\pi_1, \pi_2, \ldots$ consist of disjunctions defining *distinct* semilattices, which disjunct shall we choose when stepwise extending our semilattice as to satisfy the truth lemma?

Back to 'how to construct the satisfying semilattice model?'

- *Will the canonical model do?* ✗
- *Will 'deterministic' step-by-step do?* ✗
- *How about 'indeterministic' step-by-step?*

Failure of (deterministic) step-by-step

Final problem: If the consequents of the formulas $\pi_1, \pi_2, \ldots$ consist of disjunctions defining *distinct* semilattices, which disjunct shall we choose when stepwise extending our semilattice as to satisfy the truth lemma?

Back to 'how to construct the satisfying semilattice model?'

- *Will the canonical model do?* ✗
- *Will 'deterministic' step-by-step do?* ✗
- *How about 'indeterministic' step-by-step?*

Success of (indeterministic) step-by-step

Three ways to completeness:

Henkin (e.g., $\mathbb{K}$)

Standard step-by-step (e.g., MIL_{Pre})

'Indeterministic step-by-step' (MIL_{Sem})

Success of (indeterministic) step-by-step

Three ways to completeness:

Henkin (e.g., $\mathbf{K}$)

Standard step-by-step (e.g., MIL_{Pre})

'Indeterministic step-by-step' (MIL_{Sem})

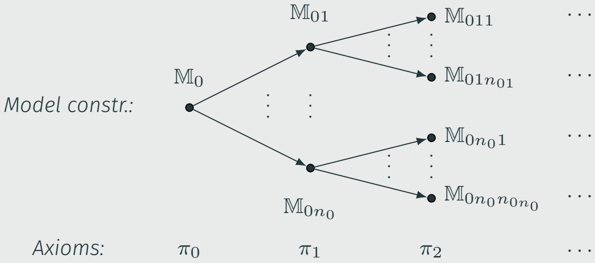
Success of (indeterministic) step-by-step

Three ways to completeness:

Henkin (e.g., $\mathbf{K}$)

Standard step-by-step (e.g., MIL_{Pre})

'Indeterministic step-by-step' (MIL_{Sem})



This completes our informal walk-through of the ideas going into the axiomatization

Conclusion

Summary and main themes:

- We went through the process of coming up with an axiomatization of MIL_{Sem}.
- Our axiomatization employed an *infinite* extension scheme.
  - This is a contrast to $MIL_{Pre} = MIL_{Pos}$;
  - and to truthmaker semantics [cf. Fine and Jago (2019)]
- Two selected take-homes:
  - Going for weak completeness facilitates 'naming' via finiteness
  - Indeterministic step-by-step when standard step-by-step fails

Open problems and further research:

- Proving (un)decidability of MIL_{Sem}.
- Applying these techniques of this talk in other settings.
- Getting clear on *why* there is this explosion in complexity from posets to semilattices; and from adding classical negation to truthmaker semantics.

Conclusion

Summary and main themes:

- We went through the process of coming up with an axiomatization of MIL_{Sem}.
- Our axiomatization employed an *infinite* extension scheme.
    - This is a contrast to $MIL_{Pre} = MIL_{Pos}$;
    - and to truthmaker semantics [cf. Fine and Jago (2019)]
- Two selected take-homes:
    - Going for weak completeness facilitates 'naming' via finiteness
    - Indeterministic step-by-step when standard step-by-step fails

Open problems and further research:

- Proving (un)decidability of MIL_{Sem}.
- Applying these techniques of this talk in other settings.
- Getting clear on *why* there is this explosion in complexity from posets to semilattices; and from adding classical negation to truthmaker semantics.

Conclusion

Summary and main themes:

- We went through the process of coming up with an axiomatization of MIL_{Sem}.
- Our axiomatization employed an *infinite* extension scheme.
    - This is a contrast to $MIL_{Pre} = MIL_{Pos}$;
    - and to truthmaker semantics [cf. Fine and Jago (2019)]
- Two selected take-homes:
    - Going for weak completeness facilitates 'naming' via finiteness
    - Indeterministic step-by-step when standard step-by-step fails

Open problems and further research:

- Proving (un)decidability of MIL_{Sem}.
- Applying these techniques of this talk in other settings.
- Getting clear on *why* there is this explosion in complexity from posets to semilattices; and from adding classical negation to truthmaker semantics.

Conclusion

Summary and main themes:

- We went through the process of coming up with an axiomatization of MIL_{Sem}.
- Our axiomatization employed an *infinite* extension scheme.
  - This is a contrast to $MIL_{Pre} = MIL_{Pos}$;
  - and to truthmaker semantics [cf. Fine and Jago (2019)]
- Two selected take-homes:
  - Going for weak completeness facilitates 'naming' via finiteness
  - Indeterministic step-by-step when standard step-by-step fails

Open problems and further research:

- Proving (un)decidability of MIL_{Sem}.
- Applying these techniques of this talk in other settings.
- Getting clear on *why* there is this explosion in complexity from posets to semilattices; and from adding classical negation to truthmaker semantics.

Conclusion

Summary and main themes:

- We went through the process of coming up with an axiomatization of MIL_{Sem}.
- Our axiomatization employed an *infinite* extension scheme.
  - This is a contrast to $MIL_{Pre} = MIL_{Pos}$;
  - and to truthmaker semantics [cf. Fine and Jago (2019)]
- Two selected take-homes:
  - Going for weak completeness facilitates 'naming' via finiteness
  - Indeterministic step-by-step when standard step-by-step fails

Open problems and further research:

- Proving (un)decidability of MIL_{Sem}.
- Applying these techniques of this talk in other settings.
- Getting clear on *why* there is this explosion in complexity from posets to semilattices; and from adding classical negation to truthmaker semantics.

Thank you!

References I

- Fine, K. and M. Jago (2019). **"Logic for Exact Entailment".** In: *The Review of Symbolic Logic* 12.3, pp. 536–556. DOI: *10.1017/S1755020318000151*.
- Knudstorp, S. B. (Forthcoming). **"Modal Information Logics: Axiomatizations and Decidability".** In: *Journal of Philosophical Logic*.
- Van Benthem, J. (1996). **"Modal Logic as a Theory of Information".** In: *Logic and Reality. Essays on the Legacy of Arthur Prior.* Ed. by J. Copeland. Clarendon Press, Oxford, pp. 135–168.
- Van Benthem, J. and N. Bezhanishvili (2022). **"Modal Structures in Groups and Vector Spaces".** In.
- Wang, X. and Y. Wang (2022). **"Tense Logics over Lattices".** In: *WoLLIC 2022*.